Each problem number is followed by an 11-tuple $(a_{10}, a_9, a_8, a_7, a_6, a_5, a_4, a_3, a_2, a_1, a_0)$, where a_k is the number of teams that scored k points on the problem.

1. Logarithmic sum. (30,1,5,0,0,0,0,6,4,1,6)

The sum is $\boxed{\sum_{n=1}^{2007} f(n) = 2007}$. Note that $\log_{32} n$ is rational if and only if n is an integral power of two, and $\log_{32} 2^r = r/5$. Now $1 \le 2^r \le 2007$ if and only if $0 \le r \le 10$, and the sum of $f(n)$ over these eleven values of n is

$$\frac{0}{5} + \frac{1}{5} + \frac{2}{5} + \cdots + \frac{10}{5} = \frac{55}{5} = 11.$$

The sum of $f(n)$ over the remaining $(2007 - 11)$ values of n is $2007 - 11$, so

$$\sum_{n=1}^{2007} f(n) = 2007.$$

2. A magic square. (46,0,0,1,0,0,0,4,1,0,1)

0	**20**	**07**
16	9	2
11	-2	18

The unique solution is shown at the right. Let x be the number in the upper left corner. Then the common sum is $x + 27$, and the lower left entry must be 11. To make the lower left to upper right diagonal sum to $x + 27$, the central entry must be $x + 9$. Then to make the sum on row 2 be $x + 27$, the center right element must be 2. To make column 2 sum to $x + 27$, the bottom center element must be -2. Now either the third row or third column tells us that the lower right corner must contain $x + 18$, and then the sum on the remaining diagonal is $3x + 27 = x + 27$, so $x = 0$.

3. The coefficient of x^2. (23,0,6,3,2,4,1,0,1,0,13)

If the coefficient of x in $P_{10}(x)$ is 2007, then the coefficient of x^2 in $P_{10}(x)$ is $\boxed{-110}$. We first show that $a = 55$. We have $P_1(x) = P_0(x-1)$, $P_2(x) = P_1(x-2) = P_0(x-3)$, and by an easy induction, in general, $P_n(x) = P_0(x - n(n+1)/2)$. In particular,

$$P_{10}(x) = P_0(x - 55) = (x - 55)^3 + a(x - 55)^2 - 1018(x - 55) + 2007,$$

and the coefficient of x is $3 \cdot 55^2 - 110a - 1018$. For this to equal 2007 we need $110a = 3 \cdot 55^2 - 1018 - 2007 = 6050$, making $a = 55$. Then the coefficient of x^2 in $P_{10}(x)$ is $-3 \cdot 55 + a = -110$.

4. Integral of fractional part. (17,1,0,7,1,4,1,4,0,1,17)

The value is $\boxed{\sqrt{2}+\sqrt{3}-7/3}$. Note that $\langle x+n\rangle = \langle x\rangle$ for every real x and every integer n, so $\langle x^2+2x-3\rangle = \langle (x+1)^2\rangle$. Using the substitution $u = x+1$, then, we have

$$\int_{-1}^{1} \langle x^2+2x-3\rangle dx = \int_{-1}^{1} \langle (x+1)^2\rangle dx = \int_{0}^{2} \langle u^2\rangle du.$$

For $0 \le u < 1$, we have $0 \le u^2 < 1$ and $\langle u^2\rangle = u^2$. For $1 \le u < \sqrt{2}$, we have $1 \le u^2 < 2$, and $\langle u^2\rangle = u^2 - 1$. For $\sqrt{2} \le u < \sqrt{3}$, $\langle u^2\rangle = u^2 - 2$, and for $\sqrt{3} \le u < 2$, $\langle u^2\rangle = u^2 - 3$. Thus

$$\begin{aligned}\int_0^2 \langle u^2\rangle du &= \int_0^1 u^2 du + \int_1^{\sqrt{2}} (u^2-1)du + \int_{\sqrt{2}}^{\sqrt{3}} (u^2-2)du + \int_{\sqrt{3}}^{2} (u^2-3)du \\ &= \int_0^2 u^2 du - 1(\sqrt{2}-1) - 2(\sqrt{3}-\sqrt{2}) - 3(2-\sqrt{3}) \\ &= \frac{8}{3} + 1 - 6 + \sqrt{2} + \sqrt{3} \\ &= \sqrt{2} + \sqrt{3} - \frac{7}{3}.\end{aligned}$$

5. Difference of squares. (26,1,0,6,0,1,0,6,1,0,12)

There are three such pairs, namely $\boxed{(1004, 1003),\ (336, 333) \text{ and } (116, 107)}$. If $2007 = a^2 - b^2$, then

$$3^2 \cdot 223 = (a+b)(a-b) = (2007)(1) = (669)(3) = (223)(9).$$

The factorization $(a+b)(a-b) = (2007)(1)$ gives $a = 1004$, $b = 1003$. The other two factorizations yield $(a, b) = (336, 333)$ and $(116, 107)$, respectively.

6. Sum of square roots. (13,0,1,10,0,1,0,1,1,3,23)

The only such pair is $\boxed{(x, y) = (223, 892)}$. For x and y positive the equation is equivalent to $x + y + 2\sqrt{xy} = 2007$. This shows that $\sqrt{xy}$ must be an integer, so if $x = m^2a$ where a has no square factors larger than 1, then y must be of the form n^2a, with $0 < m \le n$. Thus we have

$$2007 = x + y + 2\sqrt{xy} = m^2a + n^2a + 2\sqrt{m^2n^2a^2} = (m^2 + n^2 + 2mn)a = (m+n)^2a;$$

i.e., $(m+n)^2 a = 2007 = 3^2 \cdot 223 = 1^2 \cdot 2007$, which implies that $m+n = 3$ (because $m+n = 1$ is impossible), and $a = 223$. With $0 < x \leq y$, the only possibility is $m = 1$, $n = 2$, making $x = m^2 a = 223$ and $y = n^2 a = 4 \cdot 223 = 892$. As a check, we verify that

$$x + y + 2\sqrt{xy} = 223 + 4 \cdot 223 + 2\sqrt{223 \cdot 4 \cdot 223} = 9 \cdot 223 = 2007.$$

7. Difference of square roots. (6,0,0,0,0,0,0,1,0,0,46)

Yes. Upon expanding by the binomial theorem we obtain

$$(\sqrt{2007} - \sqrt{2006})^{2008} = a - b\sqrt{2007}\sqrt{2006},$$

where a and b are integers. Then

$$(\sqrt{2007} + \sqrt{2006})^{2008} = a + b\sqrt{2007}\sqrt{2006},$$

and

$$a^2 - b^2(2007)(2006) = (2007 - 2006)^{2008} = 1.$$

Let $N = a^2$. Then $N - 1 = b^2(2007)(2006)$, and

$$\sqrt{N} - \sqrt{N-1} = a - b\sqrt{2007}\sqrt{2006} = (\sqrt{2007} - \sqrt{2006})^{2008}.$$

8. Group product (7,0,0,0,0,0,0,0,3,5,38)

Consider the $n+1$ elements $1,\ a_1,\ (a_1a_2),\ (a_1a_2a_3), \ldots, (a_1a_2\cdots a_n)$. As G has just n elements, some two of these $n+1$ elements are equal. If $1 = \prod_{k=1}^{s} a_k$ for some s, we have the desired result. In the remaining case,

$$\prod_{k=1}^{t} a_k = \prod_{k=1}^{u} a_k$$

for some t and u with $1 \le t < u \le n$. In this case,

$$\prod_{k=1}^{t} a_k = \prod_{k=1}^{t} a_k \prod_{k=t+1}^{u} a_k,$$

and we have

$$1 = \prod_{k=t+1}^{u} a_k,$$

completing the proof.

9. Limit of a sequence of integrals. (1,2,0,0,2,0,0,1,4,0,43)

The limit is $\boxed{7/24}$. We will use the fact that

$$\frac{1}{(x+1)^4} < \frac{x}{x^5+1} < \frac{1}{x^4} \quad \text{for } x \geq 1, \tag{1}$$

which is evident upon clearing of fractions and expanding. It follows from (1) that

$$I_n := n^3 \int_n^{2n} \frac{dx}{(x+1)^4} < n^3 \int_n^{2n} \frac{x dx}{x^5+1} < n^3 \int_n^{2n} \frac{dx}{x^4} =: J_n$$

for every $n \geq 1$. Now

$$I_n = \frac{n^3}{3}\Big(\frac{1}{(n+1)^3} - \frac{1}{(2n+1)^3}\Big) = \frac{1}{3}\Big(\Big(\frac{n}{n+1}\Big)^3 - \Big(\frac{n}{2n+1)}\Big)^3\Big) \to \frac{1}{3}\Big(1 - \frac{1}{8}\Big) = \frac{7}{24},$$

and

$$J_n = \frac{n^3}{3}\Big(\frac{1}{n^3} - \frac{1}{(2n)^3}\Big) = \frac{1}{3}\Big(1 - \frac{1}{8}\Big) = \frac{7}{24}.$$

It follows that

$$\lim_{n\to\infty} n^3 \int_n^{2n} \frac{x dx}{x^5+1} = \frac{7}{24}.$$

10. Lattice points on a curve. (1,0,0,0,1,0,0,3,1,1,46)

Suppose (x, y) is a lattice point on the given curve. From the facts that

$$\Big(x^2 + \frac{x}{2}\Big)^2 = x^4 + x^3 + \frac{x^2}{4} = y^2 - \frac{3}{4}x^2 - x - 1 = y^2 - \frac{3}{4}\Big(x + \frac{2}{3}\Big)^2 - \frac{2}{3} < y^2,$$

and

$$(x^2 + \frac{x}{2} + 1)^2 = x^4 + x^3 + \frac{9}{4}x^2 + x + 1 = y^2 + \frac{5}{4}x^2 \geq y^2, \tag{1}$$

we conclude that

$$x^2 + \frac{x}{2} < |y| \leq x^2 + \frac{x}{2} + 1. \tag{2}$$

If x is odd, then $|y| = x^2 + (x+1)/2$ is the only integer in this interval, and

$$\begin{aligned} y^2 = \Big(x^2 + \frac{x+1}{2}\Big)^2 &= x^4 + x^3 + x^2 + \frac{x^2+2x+1}{4} \\ &= x^4 + x^3 + x^2 + x + 1 + \frac{1}{4}(x^2 - 2x - 3) = y^2 + \frac{1}{4}(x-3)(x+1). \end{aligned}$$

It follows that $(x-3)(x+1) = 0$, and so $x = 3$ or $x = -1$. This gives us the lattice points $(3, \pm 11)$ and $(-1, \pm 1)$. If x is even, then (2) implies that $|y| = x^2 + x/2 + 1$. Then (1) implies that $y^2 = y^2 + (5/4)x^2$, and therefore that $x = 0$, giving us the lattice points $(0, \pm 1)$, and so these are the only solutions.