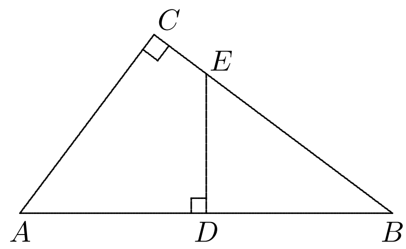
Solutions, 2004 NCS/MAA TEAM COMPETITION

Each problem number is followed by an 11-tuple $(a_{10}, a_9, a_8, a_7, a_6, a_5, a_4, a_3, a_2, a_1, a_0)$, where a_k is the number of teams that scored k points on the problem.

1. Quadrilateral area. (51,4,3,1,2,1,1,1,1,0,1)



The area is 58.5. Let h be the altitude ED. From the Pythagorean Theorem, $BC = 16$. From similar triangles, $\frac{h}{BD} = \frac{12}{16}$; i.e., $\frac{h}{10} = \frac{3}{4}$, so $h = \frac{15}{2}$. The area of triangle ABC is $\frac{1}{2}(12)(16) = 96$, and the area of triangle DBE is $\frac{1}{2}(h)(10) = \frac{75}{2}$. Then the area of the quadrilateral $ADEC$ is $96 - \frac{75}{2} = \frac{117}{2} = 58.5$.

2. Sequence sum. (39,2,5,1,4,5,4,0,1,0,5)

The sum is 0. From the recursion we find the first eight terms to be

$$a_1,\ a_2,\ a_2 - a_1,\ -a_1,\ -a_2,\ a_1 - a_2,\ a_1,\ a_2,$$

so $a_7 = a_1$ and $a_8 = a_2$. It follows that $a_{n+6} = a_n$ for all n. Let S_n be the sum of the first n terms. We see that $S_6 = 0$, and from the periodicity it follows that $S_{6n} = 0$ for all n. In particular, $S_{2004} = 0$.

3. Sum of cubes of roots . (34,3,1,1,4,10,2,0,5,0,6)

We show that

$$r^3 + s^3 = \frac{a^3 - 3a}{2}.$$

We have

$$r + s = -a \quad \text{and} \quad rs = \frac{a^2 - 1}{2}.$$

But $(r+s)^3 = r^3 + 3rs(r+s) + s^3$, so

$$\begin{aligned} r^3 + s^3 &= -3rs(r+s) + (r+s)^3 \\ &= \frac{-3(a^2-1)}{2}(-a) + (-a)^3 \\ &= \frac{a^3 - 3a}{2}. \end{aligned}$$

4. Integer linear combination. (61,0,0,0,0,0,0,0,0,0,0,5)

There do, as the example $(130)(9) + (559)(-2) = 1170 - 1118 = 52$ shows.

Divide the given equation by 13 to get the equivalent equation $10m - 4 = 43(-n)$. This tells us that we simply need $-43n$ to end in the digit 6 (strictly speaking, to be 6 mod 10). This is true, e.g., with $n = -2$, making $m = 9$.

Another approach is to use Euclid's division algorithm:

$$\begin{aligned} 559 &= 4 \cdot 130 + 39 \\ 130 &= 3 \cdot 39 + 13. \end{aligned}$$

Thus

$$\begin{aligned} 13 &= 130 - 3 \cdot 39 \\ &= 130 - 3(559 - 4 \cdot 130) \\ &= 130 - 3 \cdot 559 + 12 \cdot 130 \\ &= (130)(13) + (559)(-3). \end{aligned}$$

Multiplying by 4 we obtain the solution $(130)(52) + (559)(-12) = 52$.

5. A polynomial in x^3. (10,0,0,0,0,0,3,0,2,1,50)

Yes; one such is $Q(x) = x^6 + x^5 - 3x^3 - x^2 + 2x + 4$, making $P(x)Q(x) = x^9 - 4x^6 + 7x^3 - 8$. One finds quickly that there is no such $Q(x)$ of degree three by trying $Q(x) = x^3 + ax^2 + bx + c$. To make the coefficients of x, x^2, x^4 and x^5 in the product all equal 0 places four conditions on the three quantities a, b, c, and there is no solution. But with $Q(x) = x^6 + a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$ we have six quantities a_0, a_1, a_2, a_3, a_4, a_5 at our disposal, and six conditions to make the coefficients of x, x^2, x^4, x^5, x^7 and x^8 vanish. Denoting $P(x)Q(x)$ by $c_0 + c_1x + c_2x^2 + \cdots + c_8x^8 + x^9$, we have

$$\begin{aligned} c_1 &= a_0 - 2a_1 &&= 0 \\ c_2 &= -a_0 + a_1 - 2a_2 &&= 0 \\ c_4 &= a_1 - a_2 + a_3 - 2a_4 &&= 0 \\ c_5 &= a_2 - a_3 + a_4 - 2a_5 &&= 0 \\ c_7 &= a_4 - a_5 + 1 &&= 0 \\ c_8 &= a_5 - 1 &&= 0. \end{aligned}$$

The last two equations immediately give $a_5 = 1$ and $a_4 = 0$. Substituting these values in the equations for c_4 and c_5 and adding these two gives $a_1 = 2$. Then the equation for c_1 gives $a_0 = 4$, after which we easily find $a_2 = -1$ and $a_3 = -3$.

6. Shuffling cards. (41,0,0,2,0,0,0,1,0,0,22)

The order after one shuffle was

$$7, 6, 4, 10, 8, K, 3, Q, 5, 2, A, J, 9.$$

Let P represent the permutation executed by the shuffler. As P^2 in cycle notation is

$$(A, 3, 10, 6, 9, 8, J, 7, 4, 2, K, 5, Q),$$

a single cycle, it follows that P is a single cycle, and therefore that $P^{13} = I$. Then $P = P^{14} = (P^2)^7$, which one computes from P to be the cycle

$$(A, 7, 3, 4, 10, 2, 6, K, 9, 5, 8, Q, J).$$

This cycle applied to the cards in their original order puts them in the order

$$7, 6, 4, 10, 8, K, 3, Q, 5, 2, A, J, 9.$$

(One may also find P by filling in the middle line of the table below by trial and error:

n	A	2	3	4	5	6	7	8	9	10	J	Q	K
$P(n)$													
$P^2(n)$	3	K	10	2	Q	9	4	J	8	6	7	A	5

(1)

After trying a value for $P(A)$, the remainder of the table is determined.)

7. Slanted asymptote. (3,0,0,2,4,0,0,0,9,3,45)

The line is $y = x - 2$, as one guesses from the fact that for x negative and $|x|$ large, $\sqrt{x^2 + 4x + 5} = \sqrt{(x+2)^2 + 1}$ is approximately $|x + 2|$, which is $-x - 2$. We need to show that $|f(x) - (x - 2)| \to 0$ as $x \to -\infty$. Consider

$$\begin{aligned} |f(x) - (x-2)| &= |2x + \sqrt{(x+2)^2 + 1} - x + 2| \\ &= \left| x + 2 + \sqrt{(x+2)^2 + 1} \right| \\ &= \left| x + 2 + |x+2| \sqrt{1 + \frac{1}{(x+2)^2}} \right|. \end{aligned}$$

For $x < -2$, $|x+2| = -(x+2)$, and

$$\begin{aligned}|f(x)-(x-2)| &= \left|(x+2)-(x+2)\sqrt{1+\frac{1}{(x+2)^2}}\right| = |x+2|\left|\left(1-\sqrt{1+\frac{1}{(x+2)^2}}\right)\right| \\ &= \frac{|x+2|\left|\left(1-1-\frac{1}{(x+2)^2}\right)\right|}{1+\sqrt{1+\frac{1}{(x+2)^2}}} = \frac{1}{|x+2|\left(1+\sqrt{1+\frac{1}{(x+2)^2}}\right)} \\ &< \frac{1}{|x+2|} \to 0 \quad \text{as } x \to -\infty.\end{aligned}$$

8. Find the n-th term. (11,1,0,1,0,1,0,0,0,0,52)

The n-th term is $a_n = 3^n + 668^n$. We see that $a_0 = 2 = 3^0 + 668^0$ and $a_1 = 671 = 3^1 + 668^1$. A useful observation is that $671 = 3 + 668$ and $2004 = 3 \cdot 668$. Thus

$$a_2 = 671 \cdot 671 - 2004 \cdot 2 = (3+668)^2 - 2 \cdot 3 \cdot 668 = 3^2 + 668^2,$$

suggesting the asserted formula. To complete the proof by induction, assume that

$$a_k = 3^k + 668^k \quad \text{and} \quad a_{k+1} = 3^{k+1} + 668^{k+1}.$$

Then

$$\begin{aligned}a_{k+2} &= 671 a_{k+1} - 2004 a_k \\ &= (3+668)(3^{k+1} + 668^{k+1}) - 3 \cdot 668(3^k + 668^k) \\ &= 3^{k+2} + 668 \cdot 3^{k+1} + 3 \cdot 668^{k+1} + 668^{k+2} - 3^{k+1} \cdot 668 - 3 \cdot 668^{k+1} \\ &= 3^{k+2} + 668^{k+2},\end{aligned}$$

so by induction the claim is established.

ALTERNATE SOLUTION

We have a linear difference equation with constant coefficients. The auxiliary equation is $m^2 - 671m + 2004 = (m-3)(m-668)0$, with roots 3 and 668. Then the general solution has the form $A \cdot 3^n + B \cdot 668^n$, and the given initial values determine that $A = B = 1$.

9. Same fractonal parts. (1,0,0,0,0,2,0,0,1,3,59)

We have $x^2 - x = a$ and $x^n - x = b$, where a and b are integers. We show first that if x is rational then it is an integer. From $x^2 - x - a = 0$ we have

$$x = \frac{1 \pm \sqrt{1+4a}}{2}.$$

If this is rational then so is $\sqrt{1+4a}$, and if the square root of an integer is rational it is an integer. Thus $\sqrt{1+4a}$ is an odd integer, and x is an integer. Thus it suffices to show that x is rational. Now, $x^2 = x + a$, so

$$x^3 = x^2 + ax = (x+a) + ax = (1+a)x + a.$$

Suppose that $x^k = r_k x + s_k$ where r_k and s_k are integers. Then

$$x^{k+1} = r_k x^2 + s_k x = r_k(x+a) + s_k x = r_{k+1}x + s_{k+1},$$

where $r_{k+1} = r_k + s_k$ and $s_{k+1} = r_k a$. It follows by induction the for every positive integer m, $x^m = r_m x + s_m$ for some integers r_m and s_m. In particular, $x^n = r_n x + s_n$. This together with $x^n = x + b$ implies that $(r_n - 1)x + (s_n - b) = 0$, and thus x is rational, provided that $r_n \neq 1$.

Note that $a \geq 0$ because x is real. If $a = 0$, then $x = 0$ or 1, so is an integer. In the remaining case $a \geq 1$. Now $r_2 = 1$ and $s_2 = a \geq 1; r_3 = a + 1 > 1$ and $s_3 = a \geq 1$. Suppose $r_k > 1$ and $s_k \geq 1$. Then $r_{k+1} = r_k + s_k > r_k > 1$ and $s_{k+1} = r_k a \geq r_k > 1$. It follows by induction that $r_k > 1$ for all $k > 2$. Thus x is rational, and therefore an integer. ∎

10. Limit of product of cosines. (2,0,0,0,2,1,0,0,0,0,61)

The limit is $(\sin x)/x$. Let

$$\begin{aligned} v_n(x) &= u_n(x) \sin \frac{x}{2^n} \\ &= \cos \frac{x}{2} \cos \frac{x}{2^2} \cdots \cos \frac{x}{2^n} \sin \frac{x}{2^n}. \end{aligned}$$

Then

$$\begin{aligned} v_n(x) &= \left(\cos \frac{x}{2} \cos \frac{x}{2^2} \cdots \cos \frac{x}{2^{n-1}}\right)\left(\cos \frac{x}{2^n} \sin \frac{x}{2^n}\right) \\ &= \left(\cos \frac{x}{2} \cos \frac{x}{2^2} \cdots \cos \frac{x}{2^{n-1}}\right)\left(\frac{1}{2} \sin \frac{x}{2^{n-1}}\right) \\ &= \frac{1}{2} u_{n-1}(x) \sin \frac{x}{2^{n-1}} = \frac{1}{2} v_{n-1}(x), \end{aligned}$$

from which it follows that

$$v_n(x) = \frac{1}{2^{n-1}} v_1(x) = \frac{1}{2^{n-1}} \cos \frac{x}{2} \sin \frac{x}{2} = \frac{1}{2^n} \sin x.$$

Consequently,

$$\begin{aligned} u_n(x) &= \frac{v_n(x)}{\sin \frac{x}{2^n}} = \frac{\frac{1}{2^n} \sin x}{\sin \frac{x}{2^n}} \\ &= \frac{\frac{x}{2^n} \frac{\sin x}{x}}{\sin \frac{x}{2^n}} = \frac{\sin x}{x} \frac{\frac{x}{2^n}}{\sin \frac{x}{2^n}}. \end{aligned}$$

Since $\lim_{t \to 0} \frac{t}{\sin t} = 1$, we see that

$$\lim_{n \to \infty} u_n(x) = \frac{\sin x}{x}.$$