Solutions, 2001 NCS/MAA TEAM COMPETITION

1. A good bet? No, since the probability of at least one face card is $\frac{47}{85} > \frac{1}{2}$, as we'll show.

There are $\binom{52}{3}$ different sets of 3 cards Bertha could draw. She draws no face card if and only if she draws all 3 from the other 40 cards, which can be done in $\binom{40}{3}$ ways. Thus, the probability of no face cards is

$$\frac{\binom{40}{3}}{\binom{52}{3}} = \frac{40 \cdot 39 \cdot 38}{52 \cdot 51 \cdot 50} = \frac{38}{85},$$

so the probability of at least one face card is $1 - \frac{38}{85} = \frac{47}{85}$.

2. Cesáro sums. It is again 100. From the sums

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ s_3 &= a_1 + a_2 + a_3 \\ &\vdots \\ s_n &= a_1 + a_2 + a_3 + \cdots + a_n \end{aligned}$$

we see that

$$\frac{s_1 + s_2 + \cdots + s_n}{n} = \frac{na_1 + (n-1)a_2 + (n-2)a_3 + \cdots + a_n}{n},$$

so in our case

$$\frac{99a_1 + 98a_2 + 97a_3 + \cdots + a_{99}}{99} = 100.$$

Then the Cesáro sum of 1, a_1, a_2, $\ldots, a_{99}$ is

$$\begin{aligned} \frac{100 \cdot 1 + 99 \cdot a_1 + 98 \cdot a_2 + \cdots + a_{99}}{100} &= 1 + \frac{99a_1 + 98a_2 + \cdots + a_{99}}{100} \\ &= 1 + \frac{99}{100} \cdot \frac{99a_1 + 98a_2 + \cdots + a_{99}}{99} \\ &= 1 + \frac{99}{100} \cdot 100 \\ &= 100. \end{aligned}$$

3. An integral. The value is 4π. Let $f(x) = x\cos x^3$. Then $f(-x) = -x\cos(-x)^3 =$

$-x\cos x^3 = -f(x)$, so $\int_{-a}^{a} f(x) = 0$ for every a, and the integral reduces to

$$\int_{-\pi}^{\pi} 2dx = 4\pi.$$

4. Fractional parts. One solution is $r = 2 + \sqrt{3}$, and any number of the form $\frac{n+\sqrt{n^2-4}}{2}$

where $n \geq 3$ is an integer, is a solution. We may assume without loss of generaltity that $r > 1$, and then $0 < \frac{1}{r} < 1$. Then $F(\frac{1}{r}) = \frac{1}{r}$ and $F(r) = r - \lfloor r \rfloor$. If $F(r) + F(\frac{1}{r}) = 1$, then

$$r + \frac{1}{r} = F(r) + \lfloor r \rfloor + F\left(\frac{1}{r}\right) = 1 + \lfloor r \rfloor = n,$$

where $n \geq 2$ is an integer. Solving for r we obtain

$$r = \frac{n + \sqrt{n^2 - 4}}{2},$$

and we see that we need $n > 2$ since $n = 2$ makes $r = 1$. The case $n = 4$ gives $r = 2 + \sqrt{3}$. To verify directly that $r = 2 + \sqrt{3}$ is a solution, note that

$$F\left(\frac{1}{r}\right) = \frac{1}{r} = 2 - \sqrt{3} \quad \text{and } F(r) = \sqrt{3} - 1.$$

5. A radical limit. The limit is $\frac{1}{2}$. For,

$$\sqrt{x + \sqrt{x + \sqrt{x}}} - \sqrt{x} = \frac{\left(\sqrt{x + \sqrt{x + \sqrt{x}}} - \sqrt{x}\right)\left(\sqrt{x + \sqrt{x + \sqrt{x}}} + \sqrt{x}\right)}{\sqrt{x + \sqrt{x + \sqrt{x}}} + \sqrt{x}}$$

$$= \frac{x + \sqrt{x + \sqrt{x}} - x}{\sqrt{x + \sqrt{x + \sqrt{x}}} + \sqrt{x}} = \frac{\sqrt{1 + \sqrt{\frac{1}{x}}}}{\sqrt{1 + \sqrt{\frac{1}{x} + \sqrt{\frac{1}{x^3}}}} + \sqrt{1}}.$$

Taking the limit as $x \to \infty$ we get

$$\frac{\sqrt{1}}{\sqrt{1} + \sqrt{1}} = \frac{1}{2}.$$

6. Gaussian integers and Pythagorean triples. The assertion is correct. Here is proof:

$(a+bi)^2 = (a^2-b^2)+(2ab)i$. Now

$$\begin{aligned}(a^2-b^2)^2+(2ab)^2 &= (a^4-2a^2b^2+b^4)+4a^2b^2\\ &= a^4+2a^2b^2+b^4\\ &= (a^2+b^2)^2.\end{aligned}$$

which shows that $(|a^2-b^2|, 2ab, a^2+b^2)$ is a Pythagorian triple. (Note that this merely expresses the fact that the length of $(a+bi)^2$ is the square of the length of $a+bi$.) The fact that $ab \neq 0$ and $|a| \neq |b|$ makes the triangle nondegenerate.

7. Sum the series.

The sum is $\frac{\cosh 1+\cos 1}{2}$. Adding the Maclaurin series for $\cos x$ and $\cosh x$, we have

$$\cos x+\cosh x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}+\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!} = 2\sum_{n=0}^{\infty}\frac{x^{4n}}{(4n)!}$$

Now putting $x=1$ gives the desired result.

8. A ring with unity? We note first that if $ra=rb$ then $a=b$, since $0=ra-rb=r(a-b)$

and this implies $a-b=0$. Similarly, if $ar=br$, then $a=b$. Since R is finite, the set $\{r, r^2, r^3, \cdots\}$ is finite, so there are positive integers n and k for which $r^{n+k}=r^k$. By the cancellative property for r we may conclude that

$$r^{n+1}=r. \tag{1}$$

We show that r^n is a multiplicative identity. Given x in R, we have from (1) that $r^{n+1}x = rx$, so by the cancellative property for r, $r^n x = x$. Similarly one shows that $xr^n = x$. ■

9. A discontinuous function. Suppose, on the contrary, that f is continuous. Note that

$f(x+1)$ can never be 0 (for this would make $1 = 0$), and thus $f(x)$ is never 0. By the Intermediate Value Theorem either $f(x)$ is always positive or it is always negative. But it is clear from (1) that $f(x)$ and $f(x+1)$ cannot both be positive, so $f(x) < 0$ for all x. Then

$$0 > f(x+1) = \frac{-1}{1+f(x)}, \tag{2}$$

which implies that $1 + f(x) > 0$, so we have $0 > f(x) > -1$ for all x, and $0 < 1 + f(x) < 1$. But now (2) implies

$$f(x+1) = \frac{-1}{1+f(x)} < -1,$$

contradicting the earlier conclusion that all function values are > -1. Thus f cannot be a continuous function.

10. This year's term. We show that, if a sequence exists satisfying the given conditions,

then $a_{2001} = 2001^2$, and more generally, $a_n = n^2$ for all n. (Conversely, it is easy to show that $a_n = n^2$ satisfies the given conditions.) With $m = n = 0$ we have $2a_0 = 4a_0$, so $a_0 = 0$. With $m = 1$ and $n = 0$ we have $a_2 + a_0 = 2(a_1 + a_1) = 4$ (remember that $a_1 = 1$ is given), so $a_2 = 4$. In general, putting $n = 0$ in (1) gives $a_{2m} = 4a_m$, so $a_4 = 16$. With $m = 2$ and $n = 1$ we get $a_4 + a_2 = 2(a_3 + a_1)$, from which we easily calculate that $a_3 = 9$. By now one suspects that $a_n = n^2$ in general. We prove this by induction. Suppose that it is true for integers $\leq n$. Then $a_{2n} + a_2 = 2(a_{n+1} + a_{n-1})$, so

$$\begin{aligned} a_{n+1} &= \frac{a_{2n} + a_2}{2} - a_{n-1} \\ &= \frac{4a_n + 4}{2} - a_{n-1} \\ &= 2n^2 + 2 - (n-1)^2 \\ &= (n+1)^2. \end{aligned}$$

By the induction principle, $a_n = n^2$ for all n. ∎