THIRTEENTH ANNUAL
NORTH CENTRAL SECTION MAA
TEAM CONTEST

November 14, 2009, 9:00am to 12:00 noon

1. **Roots of a quadratic.**

   Given that one of the roots of the queation $x^2 - 2ax + m = 0$ is $a - b$, determine m in terms of a and b.

2. **Rational numbers.**

   Find the set of all real numbers x for which

   $$2x + \sqrt{4x^2 + 1} - \frac{1}{2x + \sqrt{4x^2 + 1}}$$

   is a rational number.

3. **Last integer odd.**

   One starts with the list $1, 2, 3, \ldots, 2009$. In a single move one replaces any two elements a and b of the list by $|a - b|$. E.g., one may replace the elements 1604, 122 by 1482. The list may at times contain the same integer more than once. After 2008 moves, a single integer remains. Prove that this last remaining integer is odd.

4. **Smallest C.**

   Let $f(x) = 3x^2 + Cx^{-3}$ for $x > 0$, where C is some positive constant. Find the smallest value of C such that $f(x) \geq 20$ for all $x > 0$. You must adequately defend your answer.

5. **Counting ordered quadruples of integers.**

   How many ordered quadruples of integers (a, b, c, d) are there, with $1 \leq a < b < c < d \leq 2009$, satisfying $a + d = b + c$ and $bc - ad = 2009$?

6. **Bases for F^3 over F.**

   Let F be the 5-element field of integers modulo 5, and $V = F^3$ the vector space of dimension 3 over F. Thus V may be regarded as the set of ordered triples of elements of F. A basis of V is an *unordered* linearly independent set of three vectors from V. How many different bases does V have?

7. **No consecutive heads.**

   A fair coin is tossed nine times in succession. What is the probability that there are no two consecutive heads? (The coin has two sides, one labelled heads and the other tails, each occurring with probability 1/2 each time the coin is tossed.)

8. **Chords of length $3, 5, 7$.**

   Chords of length 3, 5 and 7 in a single circle subtend angles α, β, and $\alpha + \beta$, respectively where $\alpha + \beta < \pi$. Find $\cos\alpha$.

9. **Square roots of area ratios.**

   Through a point P inside a triangle RST, three lines are drawn; one parallel to each of the sides. These lines partition RST into three triangles and three parallelograms. Let A_1, A_2, A_3 be the areas of the three inner triangles and A the area of RST. Show that

   $$\sqrt{\frac{A_1}{A}} + \sqrt{\frac{A_2}{A}} + \sqrt{\frac{A_3}{A}} = 1.$$

10. **Final digits of 2009^n.**

    Prove that there is a positive integer n such that 2009^n in decimal form ends in $000\ldots01$, where the final 1 is immediately preceded by 2009 zeros.