

NINTH ANNUAL
NORTH CENTRAL SECTION MAA
TEAM CONTEST

November 12, 2005

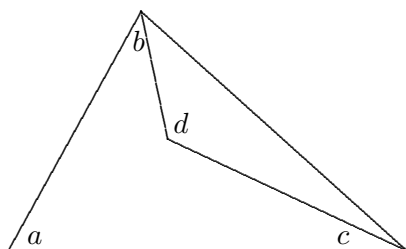
9:00 a.m. to 12:00 noon

NO BOOKS, NOTES, CALCULATORS, COMPUTERS OR NON-TEAM-MEMBERS may be consulted.

PLEASE BEGIN EACH PROBLEM ON A NEW SHEET OF PAPER. Team identification and problem number should be clearly given at the top of each sheet of paper submitted.

Each problem counts 10 points. Partial credit for significant but incomplete work. For full credit, answers must be fully justified. But in some cases this may simply mean showing all work and reasoning. Have fun!

1. Angle relationship.



In the triangle at the left, a, b, c and d are measures of angles in degrees. Find a in terms of b, c and d .

2. System of equations.

Find all real solutions (x, y) of the system

$$\begin{aligned} |x| + x + y &= 10, \\ x + |y| - y &= 12. \end{aligned}$$

Justify your answer.

3. Coefficient of x^9 .

When $\left(2x^2 - \frac{1}{x^3}\right)^{12}$ is expanded and like powers of x collected, what is the coefficient of x^9 ?

4. Final digit.

If the number $7^{(7^7)}$ is written out in decimal form, what is the last (rightmost) digit? Defend your answer.

5. A pair of integrals.

For $n = 1, 2, 3, \dots$, let

$$I_n = \int_0^{\frac{\pi}{4}} \tan^n x dx.$$

Evaluate $F(n) = I_n + I_{n+2}$.

6. Special integers.

Let a positive integer n be called “special” if the integer 1 can be expressed as a sum of n **distinct** unit fractions (i.e., fractions of the form $1/r$, where r is a positive integer). Thus, e.g., 3 is special because $1 = 1/2 + 1/3 + 1/6$. Find all special positive integers.

7. Card count.

Of two identical decks of 52 cards (it is enough to know that no two of the cards in a deck are alike), each is shuffled by itself, and one is then placed on top of the other. For each card of the top deck, one counts the number of cards lying strictly between it and the like card of the bottom deck. Find, with proof, the sum of these numbers.

8. Sum the series.

Show that

$$\sum_{n=1}^{\infty} \frac{n}{3 \cdot 5 \cdot 7 \cdots (2n+1)} = \frac{1}{2}.$$

9. Product of secants.

Prove that

$$\sec \frac{\pi}{7} \sec \frac{2\pi}{7} \sec \frac{3\pi}{7} = 8.$$

10. Powers of an irrational number.

Let $r = \sqrt{3} + \sqrt{2}$. Prove that for every positive integer n , there is a positive integer a_n satisfying

(i) $r^{2n} + r^{-2n} = 4a_n + 2$, and

(ii) $r^n = \sqrt{a_n + 1} + \sqrt{a_n}$.