

**THIRD ANNUAL**  
**NORTH CENTRAL SECTION MAA**  
**TEAM CONTEST**

November 13, 1999

9:00 a.m. to 12:00 noon

**To the team members:** These problems are meant to be fun as well as challenging. Some are more challenging than others; do not feel crushed if you are unable to work all ten of them in the allotted time. Each problem counts 10 points. Partial credit will be given for significant progress or for significant partial solutions, but a thorough job on a few of them will be better than some exploratory work on all.

**NO BOOKS, NOTES, CALCULATORS, COMPUTERS OR NON-TEAM-MEMBERS may be consulted.**

Each team may submit one solution to each problem. Think of a solution as an essay; a logical argument which makes clear why your answer to the question is correct, or why the assertion whose proof

is called for in the problem is true.

**PLEASE BEGIN EACH PROBLEM ON A NEW SHEET OF PAPER.** Team identification and problem number should be clearly given at the top of each sheet of paper submitted.

**1. Mind reading.**

You have your friend Bertha roll three dice, without showing you the outcome. Ask her to multiply the number on the first die by 5, then add 7 and double the result, add the number on the second die, multiply this result by 10, add the number on the third die, and tell you the result.

- (a) If her result is 401, what were the numbers (in order) on the three dice?
- (b) Explain how to determine in general the three numbers on the dice from the result of the prescribed calculation (and make clear why your method will work).

**2. Function iteration.**

Let  $f_1(x) = f(x) = \frac{1}{1-x}$ , and for  $n > 1$ ,  $f_n(x) = f(f_{n-1}(x))$ . Evaluate  $f_{2000}(1999)$ .

**3. Unique factorization.**

In the ring  $\mathbf{Z}[x]$  of polynomials over the ring  $\mathbf{Z}$  of integers,  $x^2 + 3x + 2 = (x + 1)(x + 2)$ , and this factorization is unique. (The factorization  $(-x - 1)(-x - 2)$  does not qualify as a different one because these factors are associates of the first ones.)

Now let  $R$  be the ring of integers mod 6; i.e.,  $R = \{0, 1, 2, 3, 4, 5\}$ , with addition and multiplication mod 6. In the polynomial ring  $R[x]$ , the factorization  $x^2 + 3x + 2 = (x + 1)(x + 2)$  is still valid. Is this

factorization unique? Justify your answer.

**4. An integral.**

Evaluate

$$\int_1^2 \frac{1}{[x^2]} dx,$$

where as usual,  $[u]$  denotes the greatest integer less than or equal to  $u$ .

**5. Every non-constant function?**

Let  $f$  be a nonconstant real-valued function defined on the set  $\mathbf{R}$  of all real numbers. Prove that there exist real numbers  $x$  and  $y$  such that

$$|f(x) - f(y)| > |x - y|^{\frac{3}{2}}.$$

**6. A double inequality.**

Given that  $a$ ,  $b$  and  $c$  are real numbers with  $a < b$  and  $a < c$ , prove that

$$a < \frac{bc - a^2}{b + c - 2a} < \min\{b, c\}.$$

**7. A system of equations.**

Consider the equations

$$\begin{aligned} \frac{1}{x} + \frac{1}{y} &= \frac{1}{z} \\ a^x &= b^y = 1999^z. \end{aligned}$$

Determine whether any solutions  $(x, y, z, a, b)$  exist in which  $x$ ,  $y$  and  $z$  are nonzero real numbers and  $a$  and  $b$  are positive integers.

**8. Least upper bound.**

Find, with proof, the smallest real number  $A$  such that the inequality

$$\frac{21}{|z^4 - 5z^2 + 6|} \leq A$$

holds for every complex number  $z$  on the circle  $|z| = 3$ .

**9. Sum the series.**

Evaluate in closed form

$$\sum_{n=0}^{\infty} \frac{\cos 3n}{n!}.$$

**10. A nonnegative function.**

Let  $p(x)$  be a polynomial with real coefficients, satisfying

$$p(x) - p'(x) - p''(x) + p'''(x) \geq 0$$

for all real  $x$ . Prove that  $p(x) \geq 0$  for all real  $x$ .