1. A good bet? Adolph shuffles a standard 52-card deck and Bertha will draw 3 cards at random from it (without replacement). Adolph offers to bet even money that at least one of the cards drawn is a face card. Should Bertha take the bet? Explain. (There are 12 face cards in the deck.)

2. Cesáro sums. If $a_1,\ a_2,\ \ldots, a_n$ is a finite sequence of numbers, its Cesáro sum is defined to be

$$\frac{s_1 + s_2 + \cdots + s_n}{n},$$

where $s_k = a_1 + a_2 + \cdots + a_k$ for each k, $1 \le k \le n$. Suppose that the Cesáro sum of the 99-term sequence $a_1,\ a_2,\ \ldots, a_{99}$ is 100. Find the Cesáro sum of the 100-term sequence

$$1,\ a_1,\ a_2,\ \ldots, a_{99}.$$

3. An integral. Evaluate

$$\int_{-\pi}^{\pi} (2 + x \cos x^3) dx,$$

and justify your answer.

4. Fractional parts. For positive numbers r, let $F(r)$ denote the fractional part of r; i.e., $F(r) = r - \lfloor r \rfloor$. Thus, e.g., $F(\frac{8}{3}) = \frac{2}{3}$. Find a positive number r such that

$$F(r) + F\left(\frac{1}{r}\right) = 1.$$

5. A radical limit. Evaluate the limit, if it exists:

$$\lim_{x\to\infty} \left(\sqrt{x + \sqrt{x + \sqrt{x}}} - \sqrt{x} \right).$$

6. Gaussian integers and Pythagorean triples. A *Gaussian integer* is a complex number $a+bi$ where a and b are real integers. For example, $2+3i$ is a Gaussian integer. Observe that $(2+3i)^2 = -5+12i$, and that 5 and 12 are the first two terms of a Pythagorean triple, (5,12,13). (That (a,b,c) is a Pythagorean triple means that a, b and c are positive integers and $a^2+b^2=c^2$.) Is this always the case? Prove or find a counter-example to the following assertion: For every Gaussian integer $a+bi$ with $|a| \neq |b|$ and $ab \neq 0$, the real and imaginary parts of $(a+bi)^2$ are, in absolute value, the first two terms of a Pythagorean triple.

7. Sum the series. As we learn in elementary calculus,

$$\sum_{n=0}^{\infty} \frac{1}{n!} = e.$$

Find the sum of the series

$$\sum_{n=0}^{\infty} \frac{1}{(4n)!}.$$

8. A ring with unity? Let R be a finite ring (not necessarily commutative) containing an element r which is not a divisor of zero. (That r is not a divisor of zero means that for every nonzero element s of R we have $rs \neq 0$ and $sr \neq 0$.) Prove that R has a multiplicative identity.

9. A discontinuous function. The funtion f is defined on $\boldsymbol{R}$ to $\boldsymbol{R}$, where $\boldsymbol{R}$ is the set of real numbers, and satisfies the equation

$$f(x+1)f(x) + f(x+1) + 1 = 0 \quad \text{for all } x. \tag{1}$$

Prove that f is not continuous.

10. This year's term. If a sequence $a_0, a_1, a_2, \ldots$ satisfies $a_1 = 1$ and

$$a_{2m} + a_{2n} = 2(a_{m+n} + a_{m-n}) \tag{1}$$

for all integers m and n with $m \geq n \geq 0.$, determine, with proof, a_{2001}.