

**FOURTH ANNUAL**  
**NORTH CENTRAL SECTION MAA**  
**TEAM CONTEST**

November 11, 2000

9:00 a.m. to 12:00 noon

**To the team members:** These problems are meant to be fun as well as challenging. Some are more challenging than others; do not feel crushed if you are unable to work all ten of them in the allotted time. Each problem counts 10 points. Partial credit will be given for significant progress or for significant partial solutions, but a thorough job on a few will be better than some exploratory work on all.

**NO BOOKS, NOTES, CALCULATORS, COMPUTERS OR NON-TEAM-MEMBERS may be consulted.**

Each team may submit one solution to each problem. Think of your solution as an essay; a logical argument which makes clear why your answer to the question is correct, or why the assertion whose proof is called for in the problem is true.

**PLEASE BEGIN EACH PROBLEM ON A NEW SHEET OF PAPER.** Team identification and problem number should be clearly given at the top of each sheet of paper submitted.

**1. Average velocity.**

A particle is moving along a straight line so that its velocity at time  $t$  is  $3t^2$ . At what time  $t$  during the interval  $0 < t < 9$  is its velocity the same as its average velocity over the entire interval?

**2. Smoking and heart disease.**

In a certain country 2% of the population have heart disease, and 60% of those with heart disease are smokers. Of those without heart disease, 20% are smokers. What fraction of smokers have heart disease?

**3. Too large a rectangle.**

A rectangle of area 257 has as its base a section of the  $x$ -axis containing  $(0,0)$ , the opposite side being above the  $x$ -axis. Prove that part of the rectangle extends above the curve  $y = 80 - x^4$ .

**4. Limit of a fraction.**

Evaluate

$$\lim_{h \rightarrow 0} \frac{\int_1^{3+h} \sin(t^2) dt - \int_1^3 \sin(t^2) dt}{h}.$$

**5. Evenly spaced roots.**

Determine  $k$  so that the equation

$$(x^2 - 1)(x^2 - 4) = k$$

has four evenly spaced nonzero real roots.

**6. Final ones.**

Observe that  $1^3 = 1$  and  $71^3 = 357911$  ends in 2 ones. Does there exist a positive integer  $n$  for which  $n^3$ , in decimal form, ends in 2000 ones?

**7. Simplify this sum.**

Find a closed form expression for

$$f(n) = \sum_{k=1}^{n^2} \frac{n - \lfloor \sqrt{k-1} \rfloor}{\sqrt{k} + \sqrt{k-1}}.$$

Here  $\lfloor u \rfloor$  denotes, as usual, the greatest integer less than or equal to  $u$ .

**8. Trigonometric inequality.**

Prove that

$$\cos x + \cos y + \sin x \sin y \leq 2 \quad \text{for all } x, y \in \mathbf{R}.$$

**9. Perimeter 6 and integral area.**

Find all right triangles with perimeter 6 units and with integral area (or prove that none exist).

**10. Bigger than 2000 for large  $n$ .**

Find an integer  $n$  for which

$$\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n^2} > 2000.$$